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Essentially generalized λ-slant Toeplitz operators

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Abstract

We introduce the notion of an essentially generalized λ-slant Toeplitz operator on the Hilbert space L^2 for a general complex number λ, via the operator equation $\lambda M_z X - X M_{z^k} = K$, K being a compact operator on L^2 and $k(\geq 2)$ being an integer. We attempt to investigate some of the properties of this operator and also study its counterpart on H^2.

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1 Introduction

The symbols $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{C}$ denote the sets of all natural numbers, integers and complex numbers respectively. The Toeplitz operators X on the Hilbert space $L^2(=L^2(\mathbb{T})$, where $\mathbb{T}$ denotes the unit circle in $\mathbb{C}$) and on the Hardy space $H^2(=H^2(\mathbb{T}))$ are characterized by the operator equations $M_z X = X M_z$ and $U^* X U = X$ respectively, where M_z denotes the bilateral shift operator on L^2 and U denotes the unilateral forward shift operator on H^2. The sets $\{e_n\}_{n\in\mathbb{Z}}$ and $\{e_n\}_{n\geq 0}$, where each e_n is a function on $\mathbb{T}$ given by $e_n(z) = z^n$, form orthonormal bases of L^2 and H^2 respectively. S. Sun [12] solved completely the operator equation $U^* X U = \lambda X$, for a general complex number λ and the solutions of this equation were referred to as λ–Toeplitz operators. In the year 1995, M.C. Ho [11] introduced the class of slant Toeplitz operators, which was further generalized to the class of k^{th}-order slant Toeplitz operators [1]. These operators are characterized as the solutions of the operator equation $M_z X = X M_{z^k}$, $k \geq 2$. The study was further extended to the operator equation $\lambda M_z X = X M_{z^k}$, for $\lambda \in \mathbb{C}$ and $k \geq 2$ and the solutions were referred to as generalized λ-slant Toeplitz operators [5].

We refer to [8] and the references therein for basic definitions and properties of the spaces L^2, H^2 and L^∞. We use the symbols $\mathcal{K}$ and $\mathcal{K}(H^2)$ to denote the set of all compact operators on L^2 and H^2 respectively. The symbols $\mathcal{B}(L^2)$ and $\mathcal{B}(H^2)$ denote the sets of all bounded linear operators on L^2 and H^2 respectively.

In a yet another important direction of study, Barría and Halmos [4] brought attention to the essential commutant of the unilateral shift (also referred to as the set of essentially Toeplitz operators). Further, Avendañ̃o [3] in the year 2002 studied Hankel operators in reference to the Calkin algebra $\mathcal{B}(H^2)/\mathcal{K}(H^2)$, thereby introducing the notion of essentially Hankel operators. The study in this direction is enhanced by introduction of many other classes of operators, like essentially λ-Hankel operators, essentially slant Toeplitz operators, essentially (λ, μ)-Hankel operators etc. (see [2], [6] and [7]).

Inspired by these various variants of Toeplitz operators and their varied applications (see [9], [13]), we are motivated to further extend this study to the class of "Essentially generalized λ-slant Toeplitz operators" on the space L^2 and also to its counterpart on the space H^2.

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2 Operators on L^2

For $k \geq 2$ and a fixed complex number λ, it is known that generalized λ-slant Toeplitz operators on L^2 are characterized as the operators satisfying the operator equation $\lambda M_z X = X M_{z^k}$ (see [5]). In fact, we have

1. If X is a solution of $\lambda M_z X = X M_{z^k}$, $|\lambda| \neq 1$, then $X = 0$.
2. For $\lambda \in \mathbb{C}$ with $|\lambda| = 1$, the operator equation $\lambda M_z X = X M_{z^k}$ admits of non-zero solutions and each non-zero solution is of the form $X = D_{\overline{\lambda}} U_\varphi$, where $D_{\overline{\lambda}}$ is the composition operator on L^2 defined as $D_{\overline{\lambda}} f(z) = f(\lambda z)$ for all $f \in L^2$ and U_φ, $\varphi \in L^\infty$ is a k^{th}-order slant Toeplitz operator.

Our focus, in this paper, is to study the class of operators on L^2 satisfying the operator equation $\lambda M_z X - X M_{z^k} \in \mathcal{K}$, for a fixed complex number λ ($\lambda \neq 0$) and $k \geq 2$. We refer to the solutions of this equation as essentially generalized λ-slant Toeplitz operators and denote the set of all essentially generalized λ-slant Toeplitz operators on L^2 by (k, λ)-$ESTO(L^2)$.

In particular for $\lambda = 1$, this set coincides with k-$ESTO(L^2)$, the set of all k^{th}-order essentially slant Toeplitz operators (see [2]) and in addition if $k = 2$, this set is same as the set $ESTO(L^2)$, the set of all essentially slant Toeplitz operators on L^2 (see [2]).

Some basic properties of the set (k, λ)-$ESTO(L^2)$ are listed below.

1. (k, λ)-$ESTO(L^2) \cap \mathcal{K} = \mathcal{K}$.
2. (k, λ)-$ESTO(L^2)$ is a norm-closed vector subspace of $\mathcal{B}(L^2)$.

It is evident that every generalized λ-slant Toeplitz operator on L^2 belongs to the set (k, λ)-$ESTO(L^2)$, though the converse is not true, as is justified by the following example.

Example 2.1. For a complex number λ with unit modulus, let T be an operator on L^2 defined as

$$Te_n = \begin{cases} e_1 & \text{if } n = 0 \\ \lambda^m e_m & \text{if } n = km - 1 \text{ for some m} \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}.$$

Let $D_{\overline{\lambda}}$ be the composition operator on L^2 defined as $D_{\overline{\lambda}} f(z) = f(\lambda z)$ for all $f \in L^2$, W_k be defined on L^2 as

$$W_k e_n = \begin{cases} e_m & \text{if } n = km \text{ for some m} \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$$

and K be defined on L^2 as

$$Ke_n = \begin{cases} e_1 & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}.$$

Then, it is easy to see that $T = D_{\overline{\lambda}} W_k M_z + K$. Hence,

$$\begin{aligned} \lambda M_z T - T M_{z^k} &= \lambda (M_z D_{\overline{\lambda}}) W_k M_z - D_{\overline{\lambda}} W_k M_{z^{k+1}} + K_1 \\ &= |\lambda|^2 D_{\overline{\lambda}} (M_z W_k) M_z - D_{\overline{\lambda}} W_k M_{z^{k+1}} + K_1 \\ &= D_{\overline{\lambda}} W_k M_{z^{k+1}} - D_{\overline{\lambda}} W_k M_{z^{k+1}} + K_1 \\ &= K_1, \end{aligned}$$

where $0 \neq K_1 = \lambda M_z K - K M_{z^k} \in \mathcal{K}$. Therefore, we can conclude that T is an essentially generalized λ-slant Toeplitz operator on L^2 which is not a generalized λ-slant Toeplitz operator.

This ensures that the set of all generalized λ-slant Toeplitz operators on L^2 is contained properly in the set (k,λ)-$ESTO(L^2)$.

Also, one can assert that the set (k,λ)-$ESTO(L^2)$ is a proper superset of $\mathcal{K}$ since T is a non-compact operator on L^2.

We now try to determine the intersection of two classes of essentially generalized λ-slant Toeplitz operators.

Theorem 2.2. Let λ and μ be complex numbers such that $\lambda \neq \mu$ and $k_1 \neq k_2$ (both are integers, ≥ 2). Then, the intersection of each pair of sets listed below is $\mathcal{K}$.

1. (k,λ)-$ESTO(L^2)$ and (k,μ)-$ESTO(L^2)$.
2. (k_1,λ)-$ESTO(L^2)$ and (k_2,μ)-$ESTO(L^2)$, where $|\lambda| \neq |\mu|$.

Proof. To prove (1), let $T \in (k,\lambda)$-$ESTO(L^2) \cap (k,\mu)$-$ESTO(L^2)$. Then $\lambda M_z T - T M_{z^k}$ and $\mu M_z T - T M_{z^k}$ are both compact operators on L^2. Therefore $(\lambda-\mu) M_z T$ is a compact operator which implies that T is a compact operator since $\lambda \neq \mu$. Hence (k,λ)-$ESTO(L^2) \cap (k,\mu)$-$ESTO(L^2)$ $\subseteq \mathcal{K}$. The converse inclusion is trivial.
For the proof of (2), since $k_1 \neq k_2$, assume that $k_1 < k_2$ (If $k_1 > k_2$, we obtain the same result by working in a similar manner). Let $T \in (k_1,\lambda)$-$ESTO(L^2) \cap (k_2,\mu)$-$ESTO(L^2)$. This implies that the operator $\lambda M_z T \left(M_{z^{k_2-k_1}} - \frac{\mu}{\lambda} I\right)$ is a compact operator on L^2. Since $|\lambda| \neq |\mu|$ and $\sigma(M_{z^m}) = \mathbb{T}$, for any positive integer m, this implies that T is compact. Converse holds trivially.

Q.E.D.

Corollary 2.3. k-$ESTO(L^2) \cap (k,\lambda)$-$ESTO(L^2) = \mathcal{K}$, $\lambda \neq 1$.

For $\lambda = 1$, the set (k,λ)-$ESTO(L^2)$ is neither an algebra nor self–adjoint (see [2]). We try to investigate now whether (k,λ)-$ESTO(L^2)$, $\lambda \neq 1$, in general, is an algebra or a self–adjoint set. The following example helps us to ascertain.

Example 2.4. Let $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. Consider the operator T on L^2 defined as $T = D_{\overline{\lambda}} W_k M_z + K$, where $D_{\overline{\lambda}}$, W_k and K are as defined in Example 2.1. It was proved that $T \in (k,\lambda)$-$ESTO(L^2)$. We claim that $T^2 \notin (k,\lambda)$-$ESTO(L^2)$, since in order that T^2 lies in this set, the operator $\left(\lambda M_z T^2 - T^2 M_{z^k}\right)$ must be a compact operator on L^2. This implies that the operator

$$\left(\lambda M_z \left(D_{\overline{\lambda}} W_k M_z\right)^2 - \left(D_{\overline{\lambda}} W_k M_z\right)^2 M_{z^k}\right)$$

must be a compact operator on L^2, but we have

$$\left(\lambda M_z \left(D_{\overline{\lambda}} W_k M_z\right)^2 - \left(D_{\overline{\lambda}} W_k M_z\right)^2 M_{z^k}\right) e_n =$$

$$\begin{cases} \lambda^{p(k+1)} e_{p+1} & \text{if } n = k^2 p - k - 1 \text{ for some p} \in \mathbb{Z} \\ -\lambda^{p(k+1)-1} e_p & \text{if } n = k^2 p - 2k - 1 \text{ for some p} \in \mathbb{Z}\ , \\ 0 & \text{otherwise} \end{cases}$$

which contradicts the compactness of this operator. Thus, the set (k,λ)-$ESTO(L^2)$, $|\lambda|=1$ is not an algebra.

Further, the set is not self–adjoint since $T^* \notin (k,\lambda)$-$ESTO(L^2)$. We have $T^* = M_{\overline{z}}{W_k}^* D_\lambda + K^*$ and simple computations yield that $(\lambda M_z(M_{\overline{z}}{W_k}^*D_\lambda) - (M_{\overline{z}}{W_k}^*D_\lambda)M_{z^k})\, e_n = \overline{\lambda}^{n-1} e_{kn} - \overline{\lambda}^{n+k} e_{kn+k^2-1}$ for each $n \in \mathbb{Z}$. This helps to provide that $(\lambda M_z T^* - T^* M_{z^k})$ is a non–compact operator on L^2.

We now attempt to investigate the condition which ensures that the product of two essentially generalized λ-slant Toeplitz operators is again an essentially generalized λ-slant Toeplitz operator.

Theorem 2.5. If $T_1, T_2 \in (k,\lambda)$-$ESTO(L^2)$, then $T_1T_2 \in (k,\lambda)$-$ESTO\ (L^2)$ if and only if $T_1M_{z^k}T_2 - \lambda T_1 M_z T_2 \in \mathcal{K}$.

Proof. Let $T_1, T_2 \in (k,\lambda)$-$ESTO(L^2)$. Then,

$$\lambda M_z\,(T_1T_2) - (T_1T_2)\,M_{z^k} = (T_1M_{z^k}T_2 - T_1T_2M_{z^k})(mod\ \mathcal{K}) = (T_1M_{z^k}T_2 - \lambda T_1M_zT_2)(mod\ \mathcal{K}).$$

Hence the result. Q.E.D.

Once we put $\lambda = 1$, we can draw the conclusion that the product of two k^{th}-order essentially slant Toeplitz operators is again a k^{th}-order essentially slant Toeplitz operator if and only if $T_1M_{z^k}T_2 = T_1M_zT_2(mod\ \mathcal{K})$, which is also proved in [2].

For a natural number $p > 1$, let $n(p)$ denotes the number of partitions of p as a sum of two natural numbers. Then, for each $1 \le i \le n(p)$, we have a partition of p, say, $p = m_i + n_i$; $m_i, n_i \in \mathbb{N}$. The following theorem now follows without any extra efforts.

Theorem 2.6. Let $T \in (k,\lambda)$-$ESTO(L^2)$ and $p \in \mathbb{N}$, $p > 1$. If $T^{m_i}, T^{n_i} \in (k,\lambda)$-$ESTO(L^2)$ and $p = m_i + n_i$; $m_i, n_i \in \mathbb{N}$ for $1 \le i \le n(p)$, then the following are equivalent.

1. $T^p \in (k,\lambda)$-$ESTO(L^2)$.
2. $T^{m_i}M_{z^k}T^{n_i} = \lambda T^{m_i}M_zT^{n_i}(mod\ \mathcal{K})$.
3. $T^{n_i}M_{z^k}T^{m_i} = \lambda T^{n_i}M_zT^{m_i}(mod\ \mathcal{K})$.

Making use of the fact that for $\varphi, \psi \in L^\infty$, $M_\varphi M_\psi = M_{\varphi\psi}$ and using recursively the definition of an essentially generalized λ-slant Toeplitz operator, we obtain the following theorem.

Theorem 2.7. Let k_1, k_2 (both ≥ 2) be integers and λ be a complex number. If $T_1 \in (k_1,\lambda)$-$ESTO(L^2)$ and $T_2 \in (k_2,\lambda)$-$ESTO(L^2)$, then the following are equivalent.

1. $T_1T_2 \in (k_1k_2,\lambda)$-$ESTO(L^2)$.
2. $\left(1-\lambda^{k_1}\right) T_1M_{z^{k_1}}T_2 \in \mathcal{K}$.

In addition, if $\lambda \neq 0$, then (1) and (2) are equivalent to the condition $\left(1-\lambda^{k_1}\right) T_1T_2M_{z^{k_1k_2}} \in \mathcal{K}$.

Some immediate observations from the above theorem are:

(i) If λ is a k_1^{th}-root of unity, then the product T_1T_2 of the operators $T_1 \in (k_1, \lambda)$-$ESTO(L^2)$ and $T_2 \in (k_2, \lambda)$-$ESTO(L^2)$ is an operator in the set (k_1k_2, λ)-$ESTO(L^2)$.

(ii) The product of a k_1^{th}-order essentially slant Toeplitz operator and a k_2^{th}-order essentially slant Toeplitz operator is an essentially slant Toeplitz operator of $(k_1k_2)^{th}$-order.

Let us try to illustrate observation (i) in light of the following example. Let λ be a third root of unity i.e. λ is either 1, ω or ω^2, where $\omega = \frac{-1+\sqrt{3}i}{2}$. Consider the operators T_1 and T_2 on L^2 defined as $T_1 = D_{\overline{\lambda}}W_3M_z + K$ and $T_2 = D_{\overline{\lambda}}WM_z + K$, where $D_{\overline{\lambda}}$, W_3, $W(= W_2)$ and K are as defined in Example 2.1. Then, it is easy to see that $T_1 \in (3, \lambda)$-$ESTO(L^2)$ and $T_2 \in (2, \lambda)$-$ESTO(L^2)$. Now, the operators T_1T_2 and T_2T_1 are given as

$$T_1T_2e_n = \begin{cases} e_1 & \text{if } n = -1 \\ \lambda^{m-1}e_m & \text{if n=6m-3 for some m} \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$$

and

$$T_2T_1e_n = \begin{cases} \lambda e_1 & \text{if } n = 0 \\ \lambda^2 e_m & \text{if n=6m-4 for some m} \in \mathbb{Z}\ . \\ 0 & \text{otherwise} \end{cases}$$

The matrix representation of the operator T_1T_2 w.r.t orthonormal basis $\{e_n\}_{n\in\mathbb{Z}}$ is given as follows

$$\left[\begin{array}{cccccccccc} & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & A & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \hline \cdots & 0 & A & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 1 & 0 & 0 & 0 & A & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{array}\right],$$

where A is a (3 x 13) matrix with all columns having zero entries except the first, seventh and thirteenth columns, which are $(1, 0, 0)^t$, $(0, \lambda, 0)^t$ and $(0, 0, \lambda^2)^t$ respectively. Here, $(\cdot)^t$ denotes the transpose of matrix $(\cdot)$.

Similarly, one can also obtain the matrix representation of the operator T_2T_1. Using Theorem 2.7, we conclude that both T_1T_2 and T_2T_1 belong to the set $(6, \lambda)$-$ESTO(L^2)$.

The following theorem provides a sufficient condition so that product of any two bounded operators on L^2 lies in the set (k, λ)-$ESTO(L^2)$.

Theorem 2.8. Let $T_1, T_2 \in \mathcal{B}(L^2)$, then $T_1T_2 \in (k, \lambda)$-$ESTO(L^2)$ if any one of the following conditions holds.

1. T_1 is in essential commutant of M_z and $T_2 \in (k, \lambda)$-$ESTO(L^2)$.

2. $T_1 \in (k,\lambda)$-$ESTO(L^2)$ and T_2 is in essential commutant of M_{z^k}.

Proof. Let $T_1, T_2 \in \mathcal{B}(L^2)$ such that $T_1M_z - M_zT_1 \in \mathcal{K}$. In this case, $(\lambda M_z (T_1T_2) - (T_1T_2) M_{z^k}) = (\lambda M_zT_1T_2 - \lambda T_1M_zT_2) (mod\ \mathcal{K}) = (\lambda M_zT_1T_2 - \lambda M_zT_1T_2) (mod\ \mathcal{K}) = 0(mod\ \mathcal{K})$. Hence, $T_1T_2 \in (k,\lambda)$-$ESTO(L^2)$. Similarly, we can prove the result when condition (2) holds. Q.E.D.

Remark 2.9. If M_φ is a multiplication operator induced by $\varphi \in L^\infty$ and $T \in (k,\lambda)$-$ESTO(L^2)$, then $M_\varphi T$ and TM_φ both belong to (k,λ)-$ESTO(L^2)$.

We end this section by obtaining a necessary condition for an essentially generalized λ-slant Toeplitz operator to be self–adjoint.

Proposition 2.10. If $T, T^* \in (k,\lambda)$-$ESTO(L^2)$, then $AT^* - T^*A^* \in \mathcal{K}$, where $A = \lambda M_z + M_{\overline{z}^k}$.

Proof. Consider $AT^* - T^*A^* = (\lambda M_zT^* - T^*M_{z^k}) - (\overline{\lambda}T^*M_{\overline{z}} - M_{\overline{z}^k}T^*) = (\lambda M_zT^* - T^*M_{z^k}) - (\lambda M_zT - TM_{z^k})^*$. Hence, if $T, T^* \in (k,\lambda)$-$ESTO(L^2)$, then $AT^* - T^*A^* \in \mathcal{K}$. Q.E.D.

Theorem 2.11. A necessary condition for an essentially generalized λ-slant Toeplitz operator T to be self–adjoint is that the operator $(\lambda M_z + M_{\overline{z}^k})T$ is essentially self–adjoint.

3 Compressions

In [5], the compression of a generalized λ-slant Toeplitz operator to H^2 has been characterized as the solution X of the operator equation $\lambda X = T_{\overline{z}}XT_{z^k}$. We obtain the same characterization following the approach of [10, Problem-194] and using the matrix characterization of a generalized λ-slant Toeplitz operator.

Theorem 3.1. A bounded operator A on H^2 is the compression of a generalized λ-slant Toeplitz operator to H^2 if and only if $\lambda A = T_{\overline{z}}AT_{z^k}$, where $T_{\overline{z}}$ and T_{z^k} are Toeplitz operators on H^2 induced by $\overline{z}$ and z^k respectively.

Proof. Let A be the compression of a generalized λ-slant Toeplitz operator to H^2. Then, using the matrix characterization of A, we have that for each $i, j \geq 0$, $\lambda\langle Az^j, z^i\rangle = \langle T_{\overline{z}}AT_{z^k}z^j, z^i\rangle$ and hence $\lambda A = T_{\overline{z}}AT_{z^k}$.

Conversely, let A satisfies the given equation. Then retracing back the above steps, we obtain that $\lambda\langle Az^j, z^i\rangle = \langle Az^{j+k}, z^{i+1}\rangle$. For each non–negative integer n, consider the operator A_n on L^2 given by

$$A_n = \frac{1}{\lambda^n} S^{*n} APS^{kn},$$

where S denotes the bilateral shift on L^2 and P denotes the orthogonal projection of L^2 onto H^2 . Clearly, $||A_n|| \leq ||A||$. For each pair (i,j) of integers, we have $\langle A_nz^j, z^i\rangle = \langle \frac{1}{\lambda^n} APz^{j+kn}, z^{i+n}\rangle$. Then, for sufficiently large n ($n \geq n_0$, where n_0 is the least integer such that $j + kn_0, i + n_0 \geq 0$), we have that $|\langle A_nz^j, z^i\rangle| = \langle A_nz^j, z^i\rangle = |\frac{1}{\lambda^{n_0}}\langle Az^{j+kn_0}, z^{i+n_0}\rangle| = \langle Az^{j+kn_0}, z^{i+n_0}\rangle$.

Following the same methods and techniques as in [10], we find a bounded linear operator A_∞ on L^2 such that $\varphi(f,g) = \langle A_\infty f, g\rangle$ for all $f, g \in L^2$, which helps to provide that $\lim_{n\to\infty}\langle A_nf, g\rangle = \langle A_\infty f, g\rangle$ for all $f, g \in L^2$. Lastly, it is easy to see that A_∞ is a generalized λ-slant Toeplitz operator on L^2 and A is its compression to H^2. For, if $i, j \in \mathbb{Z}$, then

$$\langle A_nz^j, z^i\rangle = \frac{1}{\lambda}\lim_{n\to\infty}\langle \frac{1}{\lambda^n} S^{*n}APS^{kn}z^{j+k}, S^nz^{i+1}\rangle = \frac{1}{\lambda}\langle A_\infty z^{j+k}, z^{i+1}\rangle$$

and for f, g $\in H^2$, we have $\langle PA_\infty f, g\rangle = \langle A_\infty f, g\rangle = \lim_{n\to\infty} \langle A_n f, g\rangle = \langle Af, g\rangle$. Hence, we are done. Q.E.D.

Since T_z is essentially unitary, the equations $\lambda Y - T_{\overline{z}} Y T_{z^k} \in \mathcal{K}(H^2)$ and $\lambda T_z Y - Y T_{z^k} \in \mathcal{K}(H^2)$ are equivalent, for any operator Y on H^2. We shall now define the counterpart of an essentially generalized λ-slant Toeplitz operator on H^2.

Definition 3.2. An operator Y on the space H^2 is said to be essentially compression of a generalized λ-slant Toeplitz operator to H^2 if it satisfies the operator equation

$$\lambda T_z Y - Y T_{z^k} \in \mathcal{K}(H^2).$$

Let (k,λ)-$ESTO(H^2)$ denote the set of all bounded operators on H^2 which are essentially compression of generalized λ-slant Toeplitz operators to H^2.

For $\varphi \in L^\infty$, let V_φ denote the compression of a k^{th}-order slant Toeplitz operator U_φ to H^2. Then, one can readily observe that for each $f \in H^2$, $(T_z W_k) f = (P U_{z^k}) f$. This implies that $(T_z W_k)|_{H^2} = V_{z^k} = W_k T_{z^k}$.

Using this observation, it is easy to see that if T is an operator on H^2 defined as $T = D_{\overline{\lambda}} W_k T_z + K$, where $\lambda \in \mathbb{C}$ with $|\lambda| = 1$, $D_{\overline{\lambda}}$ is the composition operator on H^2 defined as $D_{\overline{\lambda}} f(z) = f(\lambda z)$ for all $f \in H^2$ and K is defined on H^2 as $Ke_0 = e_1$ and $Ke_n = 0$ if $n \geq 1$, then $\lambda T_z T - T T_{z^k} \in \mathcal{K}(H^2)$. In fact, $\lambda T_z T - T T_{z^k}$ is a non–zero compact operator on H^2. Further, $T^*, T^2 \notin (k,\lambda)$-$ESTO(H^2)$.

The following conclusions can now easily be drawn.

1. $\mathcal{K}(H^2)$ is a proper subset of (k,λ)-$ESTO(H^2)$.
2. (k,λ)-$ESTO(H^2)$ is a proper superset of the set of all compression of generalized λ-slant Toeplitz operators to H^2.
3. The set (k,λ)-$ESTO(H^2)$ is neither self–adjoint nor an algebra.

Utilizing the fact that for any positive integer m, $\sigma(T_{z^m}) = \mathbb{D}$, where $\mathbb{D}$ denotes the closed unit disc in $\mathbb{C}$, we have the following.

Proposition 3.3. Let λ, $\mu \in \mathbb{C}$ with $\lambda \neq \mu$ and $k_1 \neq k_2$. Then, (k_1,λ)-$ESTO(H^2) \cap (k_2,\mu)$-$ESTO(H^2) = \mathcal{K}(H^2)$, if either of the following holds.

1. $k_1 > k_2$ and $|\lambda| > |\mu|$.
2. $k_1 < k_2$ and $|\lambda| < |\mu|$.

Listed below are some of the properties of the set (k,λ)-$ESTO(H^2)$, which can be readily obtained by working on in a similar fashion as in the case of (k,λ)-$ESTO(L^2)$.

1. The compression of every generalized λ-slant Toeplitz operator to H^2 belongs to (k,λ)-$ESTO(H^2)$.
2. (k,λ)-$ESTO(H^2) \cap \mathcal{K}(H^2) = \mathcal{K}(H^2)$.
3. (k,λ)-$ESTO(H^2)$ is a norm-closed vector subspace of $\mathcal{B}(H^2)$.

4. For complex numbers $\lambda \neq \mu$, (k,λ)-$ESTO(H^2) \cap (k,\mu)$-$ESTO\ (H^2) = \mathcal{K}(H^2)$.
5. If $T_1, T_2 \in (k,\lambda)$-$ESTO(H^2)$, then $T_1T_2 \in (k,\lambda)$-$ESTO(H^2)$ if and only if $T_1T_{z^k}T_2 - \lambda T_1T_zT_2 \in \mathcal{K}(H^2)$.
6. For integers k_1, k_2 (both ≥ 2) and complex number λ, let $T_1 \in (k_1,\lambda)$-$ESTO(H^2)$, $T_2 \in (k_2,\lambda)$-$ESTO(H^2)$ and $k = k_1k_2$. Then $T_1T_2 \in (k,\lambda)$-$ESTO(H^2)$ if and only if $\left(1-\lambda^{k_1}\right)T_1T_{z^{k_1}}T_2 \in \mathcal{K}(H^2)$. Further, if $\lambda \neq 0$, then $T_1T_2 \in (k,\lambda)$-$ESTO(H^2)$ if and only if $\left(1-\lambda^{k_1}\right)T_1T_2T_{z^k} \in \mathcal{K}(H^2)$.
7. Let $T_1, T_2 \in \mathcal{B}(H^2)$. If T_1 is in essential commutant of T_z and $T_2 \in (k,\lambda)$-$ESTO(H^2)$ or, $T_1 \in (k,\lambda)$-$ESTO(H^2)$ and T_2 is in essential commutant of T_{z^k}, then $T_1T_2 \in (k,\lambda)$-$ESTO(H^2)$
8. If $T, T^* \in (k,\lambda)$-$ESTO(H^2)$, then $AT^* = T^*A^*(mod\ \mathcal{K}(H^2))$, where $A = \lambda T_z + T_{\bar{z}^k}$.
9. A necessary condition for an operator T $\in (k,\lambda)$-$ESTO(H^2)$ to be self–adjoint is that the operator $(\lambda T_z + T_{\bar{z}^k})T$ is essentially self–adjoint.

Next we move on to find if essentially compression of a generalized λ-slant Toeplitz operator to H^2 is an invertible operator. The answer is in negative as is justified in the following theorem.

Theorem 3.4. The set (k,λ)-$ESTO(H^2)$, $\lambda \neq 0$ doesn't contain any invertible operator.

Proof. Let $T \in (k,\lambda)$-$ESTO(H^2)$ be a Fredholm operator of index n. Then, $\lambda T_zT = TT_{z^k} + K$, for some compact operator K on H^2. The index of the operator λT_zT is n-1, while the index of $TT_{z^k} + K$ is n-k. This implies that $k = 1$ which is a contradiction. Hence the set (k,λ)-$ESTO(H^2)$ contains no Fredholm operator and in particular no invertible operator. Q.E.D.

Using the fact that the commutator of a Toeplitz operator T_φ, $\varphi \in L^\infty$ and T_{z^m} (where m is any positive integer) is a compact operator on H^2, we obtain the following result (analogous to remark 2.9).

Theorem 3.5. Let $A \in (k,\lambda)$-$ESTO(H^2)$ and T_φ be a Toeplitz operator on H^2 induced by $\varphi \in L^\infty$, then $T_\varphi A$ and AT_φ both belong to the set (k,λ)-$ESTO(H^2)$.

Proof. Let $A \in (k,\lambda)$-$ESTO(H^2)$. Consider

$$\lambda T_z(T_\varphi A) - (T_\varphi A)T_{z^k} = (T_\varphi(\lambda T_zA - AT_{z^k}))\ (mod\ \mathcal{K}(H^2)) = 0\ (mod\ \mathcal{K}(H^2)).$$

This implies that $T_\varphi A \in (k,\lambda)$-$ESTO(H^2)$. Working on similar lines, we can easily prove that AT_φ belongs to the set (k,λ)-$ESTO(H^2)$. Q.E.D.

Before we proceed further, let us recall the following definitions.

Definition 3.6. (see [4]) A bounded linear operator T on H^2 is essentially Toeplitz if $T_z^*TT_z - T \in \mathcal{K}(H^2)$. The set of all essentially Toeplitz operators is denoted by essToep.

Definition 3.7. (see [7]) A bounded linear operator T on H^2 is essentially λ-Toeplitz if $T_z^*TT_z - \lambda T \in \mathcal{K}(H^2)$. The set of all essentially λ- Toeplitz operators is denoted by essToep$_\lambda$.

In the following theorem, we describe the products of essentially Toeplitz operators and essentially $\frac{1}{\lambda}$-Toeplitz operators with the operators in the class (k,λ)-$ESTO(H^2)$. It is interesting to obtain that, in either case, the product turns out to be an operator in the class (k,λ)-$ESTO(H^2)$.

Theorem 3.8. For a complex number λ and integer $k \geq 2$, we have the following.

1. (essToep) $\big((k,\lambda)\text{-}ESTO(H^2)\big) \subseteq (k,\lambda)\text{-}ESTO(H^2)$.
2. $\left(essToep_{\frac{1}{\lambda}}\right) \big((k,\lambda^2)\text{-}ESTO(H^2)\big) \subseteq (k,\lambda)\text{-}ESTO(H^2)$.

Proof. We just prove (2). Let $T_1 \in \text{essToep}_{\frac{1}{\lambda}}$ and $T_2 \in (k,\lambda^2)$-$ESTO(H^2)$. Then, $T_zT_1 - \lambda T_1T_z \in \mathcal{K}(H^2)$ and $\lambda^2T_zT_2 - T_2T_{z^k} \in \mathcal{K}(H^2)$. Hence,

$$\begin{aligned}\lambda T_z(T_1T_2) - (T_1T_2)T_{z^k} &= \lambda^2T_1T_zT_2 - T_1T_2T_{z^k} \ \ (mod\ \mathcal{K}(H^2))\\ &= T_1(\lambda^2T_zT_2 - T_2T_{z^k})\ \ (mod\ \mathcal{K}(H^2))\\ &= 0\ \ (mod\ \mathcal{K}(H^2)).\end{aligned}$$

Therefore, $(\text{essToep}_{\frac{1}{\lambda}})\ ((k,\lambda^2)\text{-}ESTO(H^2)) \subseteq (k,\lambda)\text{-}ESTO(H^2)$. Q.E.D.

Corollary 3.9. (essToep) $\big(\text{k-}ESTO(H^2)\big) \subseteq \text{k-}ESTO(H^2)$.

Remark 3.10. It is worth mentioning here that reversing the order of composition of operators T_1 and T_2 in Theorem 3.8 (1) yields no change in the result, i.e. $\big((k,\lambda)\text{-}ESTO(H^2)\big)\,(essToep) \subseteq (k,\lambda)\text{-}ESTO(H^2)$. However, in case (2), we obtain that the product T_2T_1 of the operators $T_2 \in \big((k,\lambda^2)\text{-}ESTO(H^2)\big)$ and $T_1 \in \left(essToep_{\frac{1}{\lambda}}\right)$ belongs to the set (k,λ)-$ESTO(H^2)$ if and only if λ is a $(k-1)^{th}$-root of unity or $T_2T_{z^k}T_1 \in \mathcal{K}(H^2)$.

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